On generating functions in the AKNS hierarchy

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February 8, 2008

Abstract

It is shown that the self-induced transparency equations can be interpreted as a generating function for as positive so negative flows in the AKNS hierarchy. Mutual commutativity of these flows leads to other hierarchies of integrable equations. In particular, it is shown that stimulated Raman scattering equations generate the hierarchy of flows which include the Heisenberg model equations. This observation reveals some new relationships between known integrable equations and permits one to construct their new physically important combinations. Reductions of the AKNS hierarchy to ones with complex conjugate and real dependent variables are also discussed and the corresponding generating functions of positive and negative flows are found. Generating function of Whitham modulation equations in the AKNS hierarchy is obtained.

It is well known that many physically important integrable partial differential equations belong to the AKNS hierarchy [1]. Up to now most attention was paid to its positive flows, where both 2×2 matrices $\mathbb{U}$ and $\mathbb{V}$ have matrix elements polynomial in the spectral parameter λ what leads to the recursive structure of the hierarchy so that subsequent flows are connected by the recursion operator (see, e.g. [2]). However, negative flows in the AKNS hierarchy have not been considered systematically enough, though the sine-Gordon equation and its connection with the mKdV hierarchy has been a

recurrent theme in the soliton literature (see, e.g., [3]–[7]). Another example of well-studied negative flow is provided by the self-induced transparency (SIT) equations [8, 9] which appeared also in different forms and contexts as Pohlmeyer-Lund-Regge equations [10]–[12]. Their role as a symmetry flow in the AKNS hierarchy has been recently considered in [13] in framework of loop algebra approach to the integrable equations. Here we remark that SIT equations can be interpreted as a generating function of positive and negative flows in the AKNS hierarchy. This leads to simple method of obtaining the integrable equations of the hierarchy and corresponding Lax pairs. Generalization of this point of view leads to new useful connections between integrable equations belonging to as positive so negative flows in the AKNS hierarchy.

The AKNS hierarchy [1] is based on the Zakharov and Shabat [14] spectral problem

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{1}$$

Let us consider the first negative flow with the pole at $\lambda = \zeta$ in the complex plane of the spectral parameter λ:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_\tau = \frac{1}{\lambda - \zeta} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2}$$

The compatibility condition of systems (1) and (2) yields equations

$$\begin{aligned} &a_x = cq - br, \quad b_x - 2\zeta b = -2aq, \quad c_x + 2\zeta c = 2ar, \\ &q_\tau = -2b, \qquad r_\tau = 2c, \end{aligned} \tag{3}$$

which can be reduced to well-known SIT equations with ζ playing the role of the "detuning" parameter [8, 9]. Then, introduction of a hierarchy of times t_n labelled by inverse powers of ζ,

$$\frac{\partial}{\partial \tau} = \sum_{n=0}^{\infty} \zeta^{-n} \frac{\partial}{\partial t_n}, \tag{4}$$

and of expansions of a, b, c in inverse powers of ζ,

$$a = \sum_{n=0}^{\infty} \zeta^{-n} a_n, \quad b = \sum_{n=0}^{\infty} \zeta^{-n} b_n, \quad c = \sum_{n=0}^{\infty} \zeta^{-n} c_n, \tag{5}$$

together with geometric series expansion of $1/(\lambda-\zeta)$ in powers of λ/ζ leads at once to well-known recurrence relations for the positive AKNS hierarchy [2],

$$\begin{aligned} \sum_{n=0}^{\infty}\zeta^{-n}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t_n} &= \sum_{n=0}^{\infty}\zeta^{-n}\begin{pmatrix}A_n & B_n\\ C_n & -A_n\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ &\equiv \sum_{n=0}^{\infty}\zeta^{-n}\mathbb{V}_n\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \end{aligned} \tag{6}$$

where

$$\begin{array}{l} A_0=-a_0, \quad B_0=-b_0, \quad C_0=-c_0;\\ A_n=\lambda A_{n-1}-a_n, \quad B_n=\lambda B_{n-1}-b_n, \quad C_n=\lambda C_{n-1}-c_n, \end{array} \tag{7}$$

and a_n, b_n, c_n can be found by substitution of (4) and (5) into (3) and equating coefficients in terms with the same powers of ζ^{-1}. In this way with the choice of constants $a_0=1$, $a_1=0$ we arrive at well-known equations of the positive AKNS hierarchy

$$\begin{array}{l} q_{t_1}=-2q, \qquad r_{t_1}=2r;\\ q_{t_2}=-q_x, \qquad r_{t_2}=-r_x;\\ q_{t_3}=\frac{1}{2}(-q_{xx}+2q^2r), \quad r_{t_3}=\frac{1}{2}(r_{xx}-2r^2q);\\ q_{t_4}=\frac{1}{4}(-q_{xxx}+6qrq_x), \quad r_{t_4}=\frac{1}{4}(-r_{xxx}+6qrr_x);\ldots, \end{array} \tag{8}$$

and corresponding matrices $\mathbb{V}_n$ governing the t_n evolution of ψ function (see (6)).

In a similar way we obtain negative flows of the AKNS hierarchy, if we expand (2) and (3) in positive powers of ζ,

$$\begin{array}{l} \frac{\partial}{\partial\tau}=\sum_{n=1}^{\infty}\zeta^n\frac{\partial}{\partial t_{-n}},\\ a=\sum_{n=1}^{\infty}\zeta^n a_{-n}, \quad b=\sum_{n=1}^{\infty}\zeta^n b_{-n}, \quad c=\sum_{n=1}^{\infty}\zeta^n c_{-n}, \end{array} \tag{9}$$

that is

$$\begin{aligned} \sum_{n=1}^{\infty}\zeta^{n}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t_{-n}} &= \sum_{n=1}^{\infty}\zeta^{n}\begin{pmatrix}A_{-n} & B_{-n}\\ C_{-n} & -A_{-n}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ &\equiv \sum_{n=0}^{\infty}\zeta^{n}\mathbb{V}_{-n}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \end{aligned} \tag{10}$$

where

$$\begin{array}{l} A_{-1}=a_{-1}/\lambda, \quad B_{-1}=b_{-1}/\lambda, \quad C_{-1}=c_{-1}/\lambda;\\ A_{-n}=(A_{-n+1}+a_{-n})/\lambda, \quad B_{-n}=(B_{-n+1}+b_{-n})/\lambda,\\ C_{-n}=(C_{-n+1}+c_{-n})/\lambda, \end{array} \tag{11}$$

so that we arrive at equations of negative flows in the AKNS hierarchy,

$$(qr)_{t_{-1}} = 2a_{-1,x}, \quad q_{xt_{-1}} = 4qa_{-1}, \quad r_{xt_{-1}} = 4ra_{-1}; \tag{12}$$

$$\begin{array}{l} (qr)_{t_{-2}} = 2a_{-2,x}, \quad q_{xt_{-2}}r - qr_{xt_{-2}} = 4a_{1,x}, \\ (q_{xt_{-2}} - 4qa_{-2})_x = 8qa_{-1}, \quad (-r_{xt_{-2}} + 4ra_{-2})_x = 8ra_{-1}; \dots . \end{array} \tag{13}$$

It is known that all these flows commute with each other. Therefore we can choose any two flows and find the corresponding compatibility condition in terms of two relevant independent variables t_m and t_n. Formally, this corresponds to exclusion of x variable from equations for (x, t_m) and (x, t_n) flows by means of redefinition of the dependent variables [15]. This is possible as far as no dependent on x reduction is imposed on the t_m and t_n flows. However, this procedure is quite tedious and in practice we can avoid it by working directly with the linear systems

$$\psi_{t_m} = \mathbb{V}_m \psi, \qquad \psi_{t_n} = \mathbb{V}_n \psi. \tag{14}$$

Then we obtain equations expressed in terms of new variables given by the coefficients of λ^k in the matrix elements of $\mathbb{V}_m$ and $\mathbb{V}_n$. For example, let us consider flows arising from commutation of the system

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{-1}} = \frac{1}{\lambda} \begin{pmatrix} a_{-1} & b_{-1} \\ c_{-1} & -a_{-1} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{15}$$

with other systems corresponding to negative flows t_{-n}. In fact, with the use of the above method, we can find the generating function of these flows. To this end, we derive the compatibility conditions of (15) and (2),

$$\begin{array}{ll} a_{-1,\tau} = a_{t_{-1}}, & a_{-1,\tau} = \frac{1}{\zeta}(b_{-1}c - bc_{-1}), \\ b_{-1,\tau} = b_{t_{-1}}, & b_{-1,\tau} = \frac{2}{\zeta}(a_{-1}b - ab_{-1}), \\ c_{-1,\tau} = c_{t_{-1}}, & c_{-1,\tau} = \frac{2}{\zeta}(c_{-1}a - ca_{-1}). \end{array} \tag{16}$$

These equations lead at once to the constraints

$$a_{-1}^2 + b_{-1}c_{-1} = f_1(t_{-1}), \qquad a^2 + bc = f_2(\tau) \tag{17}$$

and after appropriate definition of dependent variables they reduce to a particular case of stimulated Raman scattering equations [16, 17]. Now, expansions of (2) and (16) with the use of (9) yield the negative flows in terms of

independent variables t_{-1} and t_{-n}. In particular, the second negative flow takes the form of the Heisenberg model equations

$$\begin{aligned} a_{-1,t_{-2}} &= \tfrac{1}{4}(b_{-1}c_{-1,t_{-1}t_{-1}} - c_{-1}b_{-1,t_{-1}t_{-1}}), \\ b_{-1,t_{-2}} &= \tfrac{1}{2}(a_{-1}b_{-1,t_{-1}t_{-1}} - b_{-1}a_{-1,t_{-1}t_{-1}}), \\ c_{-1,t_{-2}} &= \tfrac{1}{2}(a_{-1}c_{-1,t_{-1}t_{-1}} - c_{-1}a_{-1,t_{-1}t_{-1}}), \end{aligned} \tag{18}$$

and from (2) we obtain the system

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{-2}} = \begin{pmatrix} A_{-2} & B_{-2} \\ C_{-2} & -A_{-2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{19}$$

$$\begin{aligned} A_{-2} &= \tfrac{a_{-1}}{\lambda^2} + \tfrac{1}{4\lambda}(b_{-1}c_{-1,t_{-1}} - c_{-1}b_{-1,t_{-1}}), \\ B_{-2} &= \tfrac{b_{-1}}{\lambda^2} + \tfrac{1}{2\lambda}(a_{-1}b_{-1,t_{-1}} - b_{-1}a_{-1,t_{-1}}), \\ C_{-2} &= \tfrac{c_{-1}}{\lambda^2} - \tfrac{1}{2\lambda}(a_{-1}c_{-1,t_{-1}} - c_{-1}a_{-1,t_{-1}}) \end{aligned}$$

whose compatibility condition with (15) yields equations (18). In fact, these two systems have the known form [18] with the spectral parameter λ replaced by $1/\lambda$.

From this point of view, the Heisenberg equations belong to the negative part of the AKNS hierarchy. Hence, we can take a linear combination of systems (1) and (19) commuting with (15) to obtain the pair (with evident change of the notation)

$$\begin{aligned} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x &= \tfrac{1}{\lambda} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{aligned} \tag{20}$$

$$\begin{aligned} &A = \alpha\lambda + \tfrac{\beta a}{\lambda^2} + \tfrac{\beta}{4\lambda}(bc_x - cb_x), \quad B = \alpha q + \tfrac{\beta b}{\lambda^2} + \tfrac{\beta}{2\lambda}(ab_x - ba_x), \\ &C = \alpha r + \tfrac{\beta c}{\lambda^2} - \tfrac{\beta}{2\lambda}(ac_x - ca_x), \end{aligned} \tag{21}$$

where $\alpha + \beta = 1$. The compatibility condition of systems (20) yields the equations

$$\begin{aligned} a_t &= \tfrac{\beta}{4}(bc_{xx} - cb_{xx}) - \alpha(rb - qc), \\ b_t &= \tfrac{\beta}{2}(ab_{xx} - ba_{xx}) - 2\alpha qa, \\ c_t &= -\tfrac{\beta}{2}(ac_{xx} - ca_{xx}) + 2\alpha ra, \\ q_x &= -2b, \qquad r_x = 2c. \end{aligned} \tag{22}$$

Remarkably enough, this system coincides with the equations describing self-induced transparency of light propagation through medium with spatial dispersion [19]. Apparently, the fact of complete integrability of these physically important equations has not been noticed yet.

In physical applications the dependent variables are usually subject to some constraints. For example, the variables q and r may be complex (anti)conjugated to one another,

$$r = \pm q^*. \tag{23}$$

It is easy to find that the generating function of equations in the reduced AKNS hierarchy based on the Zakharov-Shabat spectral problem

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & q \\ \pm q^* & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{24}$$

is given by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_\tau = \frac{1}{\lambda - i\zeta} \begin{pmatrix} a & b \\ \mp b^* & -a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{25}$$

where ζ is considered as a real parameter. Then the compatibility conditions take the form

$$q_{x\tau} = 4aq + 2i\zeta q_\tau, \quad a_x = \pm\frac{1}{2}(|q|^2)_\tau, \quad b = -\frac{1}{2}q_\tau, \tag{26}$$

and expansions (4,5) and (9) give, respectively, positive flows,

$$\begin{aligned} &q_{t_1} - 2iq = 0, \quad q_{t_2} - q_x = 0, \quad 2iq_{t_3} - q_{xx} \pm 2|q|^2 q = 0, \\ &4q_{t_4} + q_{xxx} \pm 6|q|^2 q_x = 0, \dots , \end{aligned} \tag{27}$$

and negative flows

$$\begin{aligned} &q_{xt_{-1}} = 4a_{-1}q, \quad a_{-1,x} = \pm\tfrac{1}{2}(|q|^2)_{t_{-1}}; \\ &(q_{xt_{-2}} - 4a_{-2}q)_x = 8ia_{-1}q, \quad a_{-2,x} = \pm\tfrac{1}{2}(|q|^2)_{t_{-2}}, \\ &a_{-1,x} = \pm\tfrac{i}{4}(qq^*_{xt_{-2}} - q^* q_{xt_{-2}}); \dots . \end{aligned} \tag{28}$$

It is known (and easy to see from the above equations) that only even positive flows and odd negative flows are consistent with the reduction

$$r = \pm q \qquad (\text{or} \quad q^* = q). \tag{29}$$

These flows correspond to the mKdV-sine-Gordon hierarchy based on the spectral problem

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & q \\ \pm q & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{30}$$

It is easy to show that the generating function of this hierarchy is given by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_\tau = \frac{1}{\lambda^2 - \zeta^2} \begin{pmatrix} \lambda a & \mp\lambda b + c \\ \lambda b \pm c & -\lambda a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{31}$$

The compatibility conditions of these two systems have the form

$$a_x = 2qb, \quad b_x = \pm 2(aq - c), \quad c_x = -\zeta^2 q_\tau, \quad q_\tau = \pm 2b. \tag{32}$$

Now positive flows correspond to expansions of (31) and (32) in powers of ζ^{-2}:

$$\begin{array}{l} \frac{\partial}{\partial \tau} = \sum_{n=0}^{\infty} \zeta^{-2n} \frac{\partial}{\partial t_{2n}}, \\ a = \sum_{n=0}^{\infty} \zeta^{-2n} a_{2n}, \quad b = \sum_{n=0}^{\infty} \zeta^{-2n} b_{2n}, \quad c = \sum_{n=0}^{\infty} \zeta^{-2n} c_{2n}. \end{array} \tag{33}$$

Then we obtain from (32) the mKdV hierarchy

$$q_{t_2} = -q_x; \quad q_{t_4} = \frac{1}{4}(\pm 6q^2 q_x - q_{xxx}); \ldots . \tag{34}$$

In a similar way, negative flows correspond to expansions in powers of ζ^2:

$$\begin{array}{l} \frac{\partial}{\partial \tau} = \sum_{n=1}^{\infty} \zeta^{2n} \frac{\partial}{\partial t_{-2n}}, \\ a = \sum_{n=1}^{\infty} \zeta^{2n} a_{-2n}, \quad b = \sum_{n=1}^{\infty} \zeta^{2n} b_{-2n}, \quad c = \sum_{n=1}^{\infty} \zeta^{2n} c_{-2n}. \end{array} \tag{35}$$

Substitution of these expansions into (32) yields the systems

$$qq_{t_{-2}} = \pm a_{-2,x}, \quad \left(\frac{a_{-2,x}}{4q}\right)_x = \pm a_{-2} q; \tag{36}$$

$$\begin{array}{l} qq_{t_{-4}} = \pm a_{-4,x}, \quad \left(\frac{a_{-2,x}}{4q}\right)_x = \pm a_{-2} q, \\ \left(\frac{1}{4} q_{xt_{-4}} - q a_{-4}\right)_x = \pm \frac{4a_{-2,x}}{q}; \ldots , \end{array} \tag{37}$$

and expansion of (31) yields the corresponding t_{-2n} systems:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{-2}} = \frac{1}{\lambda} \begin{pmatrix} a_{-2} & \mp\frac{a_{-2,x}}{2q} \\ \frac{a_{-2,x}}{2q} & -a_{-2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{-4}} = \frac{1}{\lambda^4} \begin{pmatrix} A_{-4} & B_{-4} \\ C_{-4} & -A_{-4} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{39}$$

where

$$\begin{aligned} A_{-4} &= a_{-2}\lambda + a_{-4}\lambda^3, \\ B_{-4} &= qa_{-4} \mp \left(\frac{a_{-4,x}}{4q}\right)_x \mp \frac{1}{2q}(a_{-2,x}\lambda + a_{-4,x}\lambda^3), \\ C_{-4} &= \pm qa_{-4} - \left(\frac{a_{-4,x}}{4q}\right)_x + \frac{1}{2q}(a_{-2,x}\lambda + a_{-4,x}\lambda^3). \end{aligned}$$

It is easy to see that under proper normalization eqs. (36) reduce to sinh-Gordon or sine-Gordon equation depending on the choice of the sign. For definiteness we choose the lower sign and introduce

$$q = \frac{1}{2}u_x, \qquad a_{-2} = \frac{1}{2}\cos u. \tag{40}$$

Then eqs. (36) reduce to the sine-Gordon equation

$$u_{xt_{-2}} = 2\sin u. \tag{41}$$

In a similar way the second negative flow (37) takes the form

$$u_x u_{xt_{-4}} = -4a_{-4,x}, \quad \left(\frac{1}{4}u_{xxt_{-4}} - u_x a_{-4}\right)_x = 2\sin u. \tag{42}$$

Apparently, these equations has not been written down explicitly so far.

Note that generating equations (32) can be written in compact form after substitution of $q = \frac{1}{2}u_x$ and exclusion of a, b, c variables:

$$\left(\frac{u_{xx\tau} - 4\zeta^2 u_\tau}{u_x}\right)_x = \pm u_x u_{x\tau}. \tag{43}$$

It is easy to check that the first two terms of the expansion of this equation (with choice of lower sign) yield equations equivalent to (41) and (42). The next higher negative flows can be written down explicitly by this method without any difficulties.

At last, we remark that the above approach can be applied to derivation of the generating function for the Whitham modulation equations in the AKNS hierarchy. In a finite gap method of obtaining periodic solution of integrable equations, a one-phase solution of equations belonging to the AKNS hierarchy is usually parameterized by four zeros λ_i, $i = 1, 2, 3, 4$, of the fourth degree polynomial (see, e.g. [20])

$$P(\lambda) = \prod_{i=1}^{4} (\lambda - \lambda_i). \tag{44}$$

The wavelength of the periodic solution is given by the expression

$$L = \frac{1}{2} \oint \frac{d\mu}{\sqrt{P(\mu)}}. \tag{45}$$

In a modulated periodic wave the parameters λ_i become slow functions of the space and time variables and their evolution is governed by the Whitham modulation equations. Derivation of the generating function of these equation actually coincides with the derivation of the Whitham equations for the SIT system [20, 21] and therefore we shall omit here the details. The result for the generating function of the Whitham equations in the AKNS hierarchy has the form

$$\frac{\partial \lambda_i}{\partial \tau} + v_i(\lambda) \frac{\partial \lambda_i}{\partial x} = 0, \tag{46}$$

where

$$v_i(\lambda) = \left(1 - \frac{L}{\partial_i L} \partial_i\right) \frac{1}{\sqrt{P(\zeta)}}, \quad \partial_i \equiv \frac{\partial}{\partial \lambda_i}, \quad i = 1, 2, 3, 4. \tag{47}$$

Introduction of time variables according to (4) or (9) and expansion of (47) in powers of $1/\zeta$ or ζ, respectively, yields the Whitham equations for λ_i corresponding to each member of the hierarchy. The particular cases of (47) were discussed in [22] (for the KdV equations case, when $P(\lambda)$ has three zeros λ_i, $i = 1, 2, 3$) and [23] (for the NLS equation case).

In conclusion, we have found generating functions for positive and negative flows belonging to the AKNS hierarchy and some its reductions. This gives a simple method of derivation of integrable equations, their Lax pairs and Whitham modulation equations. Some known equations usually not considered as members of the AKNS hierarchy find their natural place within the AKNS scheme in our approach. This observation permits one to combine known equations into new integrable equations of physical importance.

Acknowledgements

AMK is grateful to J.F. Gomes and A.H. Zimerman for interesting discussions of negative flows in the AKNS hierarchy. This work was partially supported by RFBR (grants 00–01–00210, 00–01–00366, 01–01–00696).

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